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Design of Guaranteed Performance Controllers for Systems with Varying Parameters

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I. Introduction

SYSTEMS are designed to operate within a nominal domain that may cover different stages of a standard operation. Therefore, multiple models, or models with varying parameters, characteristic of the current operating conditions, must be established to represent the dynamics. However, the number of models and related control laws must be reduced to be tractable.

The problem of the design of guaranteed cost control laws has been a topic of interest since Chang and Peng[1] introduced the idea of modifying the Riccati equation of the standard linear quadratic regulator (LQR) problem to cope with parameter uncertainties. More recently, with the large emphasis in robust control theory, the topic has gained new interest with authors such as Vinkler and Wood,[2] Petersen and Hollot,[3] and Schmitendorf.[4]

In this Note, the results of Vinkler are extended to the case of a variable control matrix, and a new formulation of the modified Riccati equation is proposed. Guaranteed performance and stability domains are then derived around each reference point subject to such control laws. A paving of the whole operations domain is then possible using repetitive calculations. An heuristic approach is proposed to select a limited number of reference points. This approach is applied to the design of multiple laws for the longitudinal control of an airplane within its flight domain.

II. Guaranteed Cost Control Law

Let us consider the linear system given by

$$\dot{x} = A(p)x + B(p)u \qquad x \in \Re^n, \qquad u \in \Re^m \tag{1}$$

with

$$A(p) = A_0 + \sum_{i=1}^{N} p_i A_i \qquad B(p) = B_0 + \sum_{i=1}^{N} p_i B_i \tag{2}$$

where (A_0, B_0) is controllable and p is a vector of characteristic parameters of the system operating point, and matrices A_i are supposed rank one matrices. Let D_0 be the feasible operations domain. Then we consider $p \in D_p$ where D_p is a connex subset of D_0. The cost functional C over the entire operation domain is

$$C = \int_0^T (x^t Q x + u^t R u)\, \mathrm{d}t \tag{3}$$

where Q and R are, respectively, positive semidefinite and positive definite matrices. Let $S(t)$ be the $n \times n$ symmetric matrix solution of the modified Riccati equation defined as

$$\dot{S}(t) + S(t)A_0 + A_0^t S(t) - S(t)B_0 R^{-1} B_0^t S(t) + Q + P[S(t)] = 0 \tag{4}$$

with $0 \le t \le T$ and $S(T) = 0$ where matrix $P(S)$ is a symmetric upper bound of

$$E(p,S) = S[A(p) - A_0] + [A(p) - A_0]^t S + SB_0 R^{-1} B_0^t S - SB(p)R^{-1}B^t(p)S \tag{5}$$

in the sense that

$$x^t P(S) x \ge x^t E(p,S) x \qquad \forall p \in D_p \qquad \forall x \in \Re^n \tag{6}$$

In Sec. III it will be shown how to find such an upper bound. Here the following theorem holds:

Theorem: Let $S(t)$ be the solution of the modified Riccati equation (4). Then choosing the control law $u(t) = -R^{-1}B_0^t S(t)x(t)$, the value of the cost functional C is bounded above

$$\int_0^T (x^t Q x + u^t R u)\, \mathrm{d}t \le x_0^t S(0) x_0, \qquad \forall p \in D_p \tag{7}$$

So this control law is called a guaranteed cost control law over D_p.

Note that this theorem is a generalization of Theorem 1 in Ref. 2, because here we also consider uncertainty in the control matrix B.

Proof: From Eq. (4) we get

$$\forall x \in \Re^n : x^t[\dot{S} + SA_0 + A_0^t S - SB_0 R^{-1} B_0^t S + Q + P(S)]x = 0 \tag{8}$$

and replacing $P(S)$ by $E(p,S)$, the following inequality is obtained:

$$\forall x \in \Re^n : x^t[\dot{S} + SA(p) + A^t(p)S - SB(p)R^{-1}B^t(p)S + Q]x \le 0 \tag{9}$$

From Eq. (1) we get

$$\frac{\mathrm{d}}{\mathrm{d}t}(x^t S x) = x^t[\dot{S} + SA(p) + A^t(p)S]x + x^t[SB(p)]u + u^t[B^t(p)S]x \tag{10}$$

and from Eq. (8)

$$\forall x \in \Re^n : x^t[\dot{S} + SA(p) + A^t S]x \le x^t[SB(p)R^{-1}B^t(p)S - Q]x \tag{11}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}(x^t S x) \le x^t[SB(p)R^{-1}B^t(p)S - Q]x + x^t SB(p)u + u^t B^t S x \tag{12}$$

Received Aug. 22, 1991; revision received Aug. 4, 1992; accepted for publication Sept. 1, 1992.

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If $u(t)$ is chosen such that

$$u(t) = -R^{-1}B_0^tS(t)x(t) \tag{13}$$

Eq. (11) can be rewritten as

$$\frac{d}{dt}(x^tSx) \le x^t\{-Q + S[B(p)-B_0]R^{-1}[B(p)-B_0]^tS - SB_0R^{-1}B_0^tS\}x \tag{14}$$

This last inequality must be satisfied for every p in D_p, and in particular we have

$$\frac{d}{dt}(x^tSx) \le \min_{p\in Dp}\left(x^t\{-Q + S[B(p)-B_0]R^{-1}[B(p)-B_0]^tS - SB_0R^{-1}B_0^tS\}x\right) \tag{15}$$

Let

$$\tilde{Q}(p) = Q - S[B(p)-B_0]R^{-1}[B(p)-B_0]^tS + SB_0R^{-1}B_0^tS$$

or

$$\tilde{Q}(p) = Q - S\left(\sum_j p_jB_j\right)R^{-1}\left(\sum_j p_jB_j\right)^tS + SB_0R^{-1}B_0^tS$$

so that

$$x^t\tilde{Q}x = x^tQx + x^t(SB_0R^{-1}B_0^tS)x - z^tR^{-1}z$$

with

$$z = \left(\sum_j p_jB_j\right)Sx$$

So

$$\max_{p\in D_p} x^t\tilde{Q}(p)x = x^t\tilde{Q}(0)x \qquad \forall x \in \mathbb{R}^n \tag{16}$$

then

$$\forall x \in \mathbb{R}^n: -\frac{d}{dt}(x^tSx) \ge x^t(Q + SB_0R^{-1}B_0^tS)x \tag{17}$$

or

$$\forall x \in \mathbb{R}^n: -\frac{d}{dt}(x^tSx) \ge x^tQx + u^tRu \tag{18}$$

Integrating the two sides of this equation between 0 and T we get

$$\int_0^T (x^tQx + u^tRu)\,dt \le x_0^tS(0)x_0 \tag{19}$$

So we get the preceding theorem, which is a generalization of Theorem 1 in Ref. 2.

III. Modified Riccati Equation

To solve the modified Riccati equation, an upper bound of $E(p,S)$ must be taken. $E(p,S)$ can be rewritten as

$$E(p,S) = \sum_i p_i(SA_i + A_i^tS) - \sum_i p_i(SB_0R^{-1}B_i^tS + SB_iR^{-1}B_0^tS) - \sum_i\sum_{j>i} p_ip_j(SB_iR^{-1}B_j^tS + SB_jR^{-1}B_i^tS) - \sum_i p_i^2(SB_iR^{-1}B_i^tS) \tag{20}$$

The rank one A_i matrices can be rewritten as

$$A_i = v_iw_i^t, \quad i = 1,\dots,N \quad v_i \in \mathbb{R}^n \quad w_i \in \mathbb{R}^n \tag{21}$$

Now

$$x^t\left[\sum_i p_i(SA_i + A_i^tS)\right]x = x^tS\left(\sum_i p_iv_iw_i^t\right)x + x^t\left(\sum_i p_iw_iv_i^t\right)Sx \tag{22}$$

and

$$x^t\left[\sum_i p_i(SA_i + A_i^tS)\right]x \le a\sum_i |x^tSv_iw_i^tx| + \sum_i |x^tw_iv_i^tSx| \tag{23}$$

with

$$a = \max_{p_i\in D_p} |p_i| \tag{24}$$

then

$$x^t\left[\sum_i p_i(SA_i + A_i^tS)\right]x \le a\left[\sum_i (x^tSv_i)^2 + \sum_i (w_i^tx)^2\right] \tag{25}$$

or

$$x^t\left[\sum_i p_i(SA_i + A_i^tS)\right]x \le a[x^t(SVS)x + x^tWx], \quad \forall x \in \mathbb{R}^n \tag{26}$$

with

$$V = \sum_i v_iv_i^t \qquad W = \sum_i w_iw_i^t$$

Also

$$\begin{aligned}
&-(SB_iR^{-1}B_j^tS + SB_jR^{-1}B_i^tS)\\
&= S(B_i - B_j)R^{-1}(B_i - B_j)^tS - SB_iR^{-1}B_i^tS\\
&- SB_jR^{-1}B_j^tS - x^t(SB_iR^{-1}B_j^tS + SB_jR^{-1}B_i^tS)x\\
&\le x^t[S(B_i - B_j)R^{-1}(B_i - B_j)^tS]x, \qquad \forall x \in \mathbb{R}^n
\end{aligned}$$

and

$$\begin{aligned}
&-x^t(SB_0R^{-1}B_i^tS + SB_iR^{-1}B_0^tS)x\\
&\le x^t[S(B_0 - B_i)R^{-1}(B_0 - B_i)^tS]x, \qquad \forall x \in \mathbb{R}^n
\end{aligned}$$

Now we get an upper bound for $E(p,S)$

$$x^tEx \le ax\left[SVS + W + a\sum_i\sum_{j>i} S(B_i - B_j)R^{-1}(B_i - B_j)^tS\right]x \qquad \forall x \in \mathbb{R}^n \tag{27}$$

Replacing $P(S)$ by its expression in relation (4), the modified Riccati equation becomes

$$\dot{S} + S_0 + A_0^tS - SM^*S + Q^* = 0$$

with

$$S(T) = 0 \tag{28}$$

where

$$Q^* = Q + aW \tag{29}$$

$$M^* = B_0R^{-1}B_0^t - aV - a\sum_i (B_0 - B_i)R^{-1}(B_0 - B_i^t)$$

$$- a^2\sum_i \sum_{j>i} (B_i - B_j)R^{-1}(B_i - B_j)^t \tag{30}$$

It can be observed that if D_p is reduced to the nominal point of operation, the modified Riccati equation reduces to the classical one. To avoid the tendency of this approach to produce, through the solution of an LQR problem, large feedback gains resulting from the augmented Q matrix, the size of D_p must be chosen not too large. This may lead to memorize a large number of parameters relative to reference points covering the whole operations domain and to frequent changes of references during the operation.

IV. Discrete Control Structure

We consider within the total operations domain D_0 a finite set of points D_δ such that $\forall p \in D_0, \exists\, p^i \in D_\delta$ such that $\|p - p^i\| \le \delta$, where δ is a given positive distance.

The approach proposed here to get a minimal guaranteed cost control structure is composed of two procedures: first a performance mapping is realized over the points in D_δ, then a selection of a reduced number of reference points is made. The performance mapping procedure is composed of the following steps:

1) Around each point p^k of D_δ is considered a local operations domain with parameter λ: $D_{p_k}(\lambda) = \{p | p = p^k + n,\ n \in \Re^N\}$, with $|n_l/\alpha_l| \le \lambda$, $\alpha_l \ge 0$, $l = 1$ to N and $\Sigma_l \alpha_l = 1$ where the coefficients α_l are scaling factors. A bound is chosen for λ, $\lambda_k^{\max}$ such that taking $a = \max_l |n_l|$, the modified Riccati equation around point p^k has a positive semidefinite solution.

2) A max eigenvalue for $S(0)$ is assigned to each point in D_δ, Λ_k, and the following problem is solved: For each point p^k in D_δ, find the maximum value of λ_k, λ_k^* such that $0 \le \lambda_k^* \le \lambda_k^{\max}$ and such that the max eigenvalue of $S(0, \lambda_k^*)$ is less than Λ_k. Then the guaranteed cost control domain around point p^k is $\mathfrak{D}_k^c = D_{p_k}(\lambda_k^*)$.

3) Verify that δ is such that

$$\forall p^k \in D_\delta \qquad \exists p^h \in D_\delta, \qquad h \ne k$$

such that

$$p^k \in \mathfrak{D}_h^c \tag{31}$$

If this condition is not satisfied, a smaller value must be chosen for δ, and steps 1 and 2 must be run again.

When this condition is fulfilled, an overlapping structure of guaranteed cost control domains has been obtained. However, because this structure may present for practical applications too large a number of reference points, a procedure must be established to diminish their number while maintaining the control performance level. A second local domain is defined around each reference point in D_δ: Let $\lambda_k^s = \max\lambda$ such that $u_k^* = -R_k^{-1}B_0S^kx$ is stabilizing. Then $D_{p_k}(\lambda_k^s)$ is a stability domain around point p^k. If the control law u_k^* is applied to the system operating around reference point p^k, the closed-loop dynamics are given by

$$\dot{x} = (A_0^k - B_0^kR_k^{-1}B_0^{kt}S^k)x + \left[\sum_h n_h(A_h^k - B_h^kR_k^{-1}B_0^{kt}S^k)\right]x$$

or

$$\dot{x} = \tilde{A}^kx + \sum_h n_hG_h^kx \tag{32}$$

with

$$\tilde{A}^k = A_0^k - B_0^kR_k^{-1}B_0^{kt}S^k \quad \text{and} \quad G_h^k = A_h^k - B_h^kR_k^{-1}B_0^{kt}S^k \tag{33}$$

Let H^k be a positive definite solution of the Lyapunov equation

$$\tilde{A}^{kt}H + H\tilde{A}^k = -I \tag{34}$$

where I is the $n \times n$ unity matrix, and let

$$F_h^k = G_h^{kt}H + HG_h^{kt} \tag{35}$$

It has been shown[5] that a sufficient stability condition for a system governed by Eq. (32) is

$$\max_h \left|\frac{n_h}{\alpha_h}\right| \le \frac{1}{\alpha_{\max}(\Sigma_h|F_h^k|)} \tag{36}$$

So a guaranteed stability domain for system (34) around point p^k is given by $\mathfrak{D}_k^s = D_{p_k}[1/\sigma_{\max}(\Sigma_h|F_h^k|)]$. Now to each reference point p^k in D_δ it is possible to associate a guaranteed cost control domain $\mathfrak{D}_k^c$ and a guaranteed stability domain $\mathfrak{D}_k^s$. In general, it is expected that $\mathfrak{D}_k^c \subset \mathfrak{D}_k^s$. Now the procedure to delete redundant reference points in D_δ is as follows:

1) Let the initial solution set be $D^* = D_\delta$.

2) Rank the points in D_δ in decreasing order with respect to the size of their guaranteed cost control domains λ_k^*.

3) Starting from the first of these ordered points p^f, delete from the solution set any point p^h such that $p^h \in \mathfrak{D}_f^c$ and $\mathfrak{D}_h^c \subset \mathfrak{D}_f^c$.

4) Repeat step 3 until all of the remaining points in D^* have been checked. The resulting set D^* is the proposed solution set.

In the case where the system is likely to follow a reference trajectory within the operations domain, to enforce the guaranteed cost control over this trajectory, step 3 may be modified as follows: If point p^h satisfies the condition of step 3 and its deletion uncovers some part of the reference trajectory with respect to the cost control condition, it will be maintained.

V. Application: Longitudinal Control of an Aircraft

Here we consider the problem treated by Chang and Peng.[1] The simplified longitudinal motion model of a small aircraft, the Trinidad 20, is given by the equations

$$\dot{\alpha} = Z_\alpha\alpha + q + Z_\delta\delta \tag{37}$$

where α is the angle of attack

$$\dot{q} = M_\alpha\alpha + M_qq + M_\delta\delta \tag{38}$$

where q is the pitch rate and δ the elevator deflection, and

$$\dot{\theta} = q \tag{39}$$

where θ is the pitch angle and where Z_α, Z_δ, M_α, M_q, and M_δ are aerodynamic derivatives whose values are dependent on the current point in the flight envelope and evolve within the limits

$$-1.63 \le Z_\alpha \le -1.41 \qquad 0.09 \le Z_\delta \le 0.104$$

$$-8.69 \le M_\alpha \le -7.52 \qquad -1.94 \le M_q \le -1.68$$

$$-9.34 \le M_\delta \le -8.09$$

Choosing the reference point $Z_\alpha = -1.63$, $Z_\delta = 0.09$, $M_\alpha = -8.69$, $M_q = -1.94$, and $M_\delta = -8.09$, we get the representation

$$\dot{x} = (A_0 + p_1A_1 + p_2A_2 + p_3A_3)x + (B_0 + p_4B_1 + p_5B_2)\delta$$

with

$$x^t = (\alpha, q, \theta) \tag{40}$$

with

$$A_0 = \begin{bmatrix} -1.63 & 1.0 & 0.0 \\ -8.69 & -1.94 & 0.0 \\ 0.0 & 1.0 & 0.0 \end{bmatrix} \qquad A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad A_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$B_0 = \begin{bmatrix} 0.09 \\ -8.09 \\ 0.0 \end{bmatrix} \qquad B_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad B_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

The following cost function has been considered:

$$C = \int_0^{+\infty} (\alpha^2 + q^2 + r\delta^2)\,\mathrm{d}t$$

So we have

$$V = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad W = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and $0 \le a \le 1.25$.

Solving the modified Riccati equation we get

$$\text{for} \quad \lambda_{\max} = 0 \qquad S(0) = \begin{bmatrix} 0.304 & 0.0 & 0.0 \\ 0.0 & 0.0362 & 0.0 \\ 0.0 & 0.0 & 0.0 \end{bmatrix}$$

$$\text{for} \quad \lambda_{\max} = 0.2 \qquad S(0) = \begin{bmatrix} 0.450 & 0.0 & 0.0 \\ 0.0 & 0.0453 & 0.0 \\ 0.0 & 0.0 & 0.0 \end{bmatrix}$$

$$\text{for} \quad \lambda_{\max} = 0.6 \qquad S(0) = \begin{bmatrix} 0.909 & 0.01 & 0.0 \\ 0.01 & 0.0835 & 0.0 \\ 0.0 & 0.0 & 0.0 \end{bmatrix}$$

It has been found also that only five other reference points are necessary to cover the whole operations domain when the guaranteed performance level is chosen 50% above the reference level.

VI. Conclusion

A new technique derived from the guaranteed cost control method of Chang and Peng has been developed for the closed-loop control of systems with varying parameters. This technique allows the definition of guaranteed performance and stability regions around reference points. An heuristic approach is then available for the definition of a set of reference points that conveniently cover the whole operations domain. This approach seems particularly promising for aerospace applications.

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Optimal Rocket Steering in Terms of Angular Velocity of the Primer Vector

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Introduction

LAUNCHING a rocket from the Earth's surface into orbit requires a level of performance significantly above that required to achieve orbital velocity alone, about 8 km/s. It is found that the required performance as measured by the so-called ideal-velocity gain is between 10 and 11 km/s. The shortfall is due to several losses: 1) atmospheric, 2) steering, and 3) gravity.

The first results from atmospheric drag and reduced engine efficiency (due to the exhaust having to push aside the atmosphere). Steering losses result from (possibly) conflicting demands between altitude and path-angle requirements and a need to escape the atmosphere. This phenomenon forces the initial thrust to be directed more upward than otherwise desirable. Gravity losses result from the need to counter gravity via propulsion once the support of the launch platform is lost.

This difference in ideal-velocity gain between 11 km/s vs 8 km/s is quite significant and results in a payload loss of about 50%. (The difference goes to additional propellant.) Since with current booster technology it costs many hundreds of dollars to place one kilogram into orbit, any savings in required velocity gain are welcome indeed! Such savings are realized by optimal shaping of the launch trajectory.

These basic principles were recognized as far back as Tsiolkovsky,[1] who performed elementary calculations for inclined vs vertical ascents and showed that significant savings accrue from the former. His mathematical techniques, while robust, were not sufficient to arrive at a precision optimal trajectory. Credit goes to Goddard[2,3] for first recognizing the importance of the calculus-of-variations for trajectory shaping. His analysis was directed toward vertical ascents of sounding rockets, consistent with his limited stated goal of reaching "extreme altitudes." To Oberth[4,5] we owe the term "synergistic trajectory" to denote a trajectory balanced with regard to the various performance losses—in other words, an optimal trajectory. Although he developed his ideas in some detail, his methods are not based on the calculus-of-variations.

Preliminaries

The detailed rigorous solution to the optimal rocket problem was worked out in the 1950s by Lawden[6] and others.[3,7–10]

Received June 14, 1991; revision received Oct. 5, 1992; accepted for publication Jan. 10, 1993.

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